Optimization of higher order discrete inclusions in business management

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Key words
Euler-Lagrange, multivalued, higher order, transversality

Abstract.
This paper is mainly concerned with the necessary and sufficient conditions of optimality for Cauchy problem of higher order discrete inclusions. Applying optimality conditions of problems with geometric constraints, for arbitrary higher order (say s-order) discrete inclusions optimality conditions are formulated. Also some special transversality conditions, which are peculiar to problems including higher order discrete inclusions, are formulated. Formulation of necessary and sufficient conditions both for convex and non-convex discrete inclusions are based on the apparatus of locally adjoint mappings (LAMs). Furthermore, an application of these results is demonstrated by solving the problems with third order linear discrete inclusions. In particular, for first order discrete inclusions we have the so-called von Neumann economic dynamics model

AMS Subject Classifications, 49k 20, 49k24, 49J52, 49M25, 90C31

Introduction

During the last two decades the optimal control problems described by multivalued mappings consist of one of the intensive developable areas in mathematical theory of optimal processes; (see [2]-[6],[8], [10]-[13], [16,25,26,29,30,32] and their references). The problems accompanied with the higher order discrete inclusions are more complicated due to the construction an adjoint discrete inclusions and the transversality conditions. Consequently, on the whole in literature are investigated the qualitative problems with either second order discrete or differential inclusions. Most of them have been the subjects of different mathematical competitions during the last few years.

The first viability result for second order differential inclusions were given by Haddad and Yarou [14] in the case in which the multifunction is upper semicontinuous and with convex compact values, the Cauchy problem for the infinite dimensional case and second-order differential inclusion is considered. The nonconvex case has been studied by Lupulescu [17], Ibrahim and Alkulaibi [15]. In the paper [7] the existence of solutions for initial and boundary value problems for second order impulsive functional differential inclusions in Banach spaces are investigated. Here a fixed point theorem for contraction multivalued maps due to Covitz and Nadler [9] is used. In the last decade discrete and continuous time processes with lumped and distributed parameters found wide application in the field of mathematical economics and in problems of control dynamic system optimisation and differential games [1] -[11], [13,16], [18]-[28],[31,33,34].

The present paper is devoted to one of the difficult and interesting field – optimization of higher order ordinary discrete inclusions. The posed problems and the corresponding optimality conditions are new. The paper is organized as follows.

In Section1 are given the needed facts and supplementary results from the monograph of Mahmudov [25]; Hamiltonian function H and argmaximum sets of a set-valued mapping F, the locally adjoint mapping (LAM), local tent are introduced and the Cauchy problems for higher order(s -th order) discrete inclusions are formulated.

In Section 2 the optimality problem (P_D) for posed s-th order discrete inclusions in Section 1 are reduced to the problem with finite number of geometric constraints. For such problems we use constructions of convex and nonsmooth analysis and in terms of convex upper approximation , local tents, and LAMs prove necessary and sufficient conditions for optimality.

It is obvious that this method, which is certainly of independent interest from qualitative view point, can play an important role also in numerical procedures.

As is shown in these problems, in general the higher order adjoint discrete inclusion involves an auxiliary adjoint vectors. Nevertheless in the concrete problems the same inclusion involves only the "main" sequence of vectors $\left\{x_t^*\right\}_{t=3}^{T}$. In particular, for a first order discrete inclusions we can investigate the so-called von Neumann economic dynamics model [25]. Suppose we have m technological capacity manufacture output with unit commodity intensity leads manufacture of $a_j, j=1,\dots,m$ commodity, $a_j \in \mathbb{R}^n$. Thus the number of different manufactured goods is n and under the unit commodity intensity utilization of j-th technological capacity of manufacture of i-th commodity is produced amount of $a_j^i, i=1,\dots,n$ goods. Naturally, we let $a_j^i \geq 0$. Here under the unit commodity intensity employment is emitted $b_j \in \mathbb{R}^n$ commodities. Now, if at the given instant time and in the past there are x output, then intensity λ_j of each manufacture capacity, obviously must satisfy the inequality $x > b_j\lambda_j,\ \lambda_j > 0$, where is emitted commodity vector y, satisfying the equation $y = a_j\lambda_j$.
Finally, taking the matrices A, B with columns a_j, b_j, respectively we conclude that is defined a multivalued mapping, the graph of which is a polyhedral cone

$$K = \left\{(x,y): x \geq B\lambda, y = A\lambda, \lambda \geq 0, \lambda \in \mathbb{R}^m\right\}$$

where from our above stated interpretation it follows that A, B are $n \times m$ matrices with nonnegative elements and λ is a vector with components $\lambda_j, j=1,\dots,m$. We formulate the following problem:

infimum $g(x_T, T)$ subject to $x_{t+1} \in F(x_t), t = 0,1,\dots,T-1$ (x_0 is fixed), where $g(x_T,T)$ can be interpreted as the cost of the commodity vector x_T.

§1. Necessary Concepts and Problems Statements

An auxiliary notions can be found in [25]. Let $\mathbb{R}^n$ be a n-dimensional Euclidean space, $\langle x, y\rangle$ be an inner product of elements $x, y \in \mathbb{R}^n$, (x, y) be a pair of x, y. Let $P(\mathbb{R}^n)$ be a family of subsets of $\mathbb{R}^n$. Assume that $F:(\mathbb{R}^n)^s \to P(\mathbb{R}^n)$ is a multivalued (set-valued) mapping from $(\mathbb{R}^n)^s = \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{s}$ into $P(\mathbb{R}^n)$. Then $F:(\mathbb{R}^n)^s \to P(\mathbb{R}^n)$ is convex if its graph is a convex subset of $(\mathbb{R}^n)^{s+1}$, where $\operatorname{gph} F = \{(x, v_1, ..., v_{s-1}, v_s): \ v_s \in F(x, v_1, ..., v_{s-1})\}$. The multivalued mapping F is convex closed if its graph is a convex closed set in $(\mathbb{R}^n)^{s+1}$. It is convex-valued if $F(x, v_1, ..., v_{s-1})$ is a convex set for each $(x, v_1, ..., v_{s-1}) \in \operatorname{dom} F = \{(x, v_1, ..., v_{s-1}) : F(x, v_1, ..., v_{s-1}) \neq \varnothing\}$. Let us introduce the Hamiltonian function and argmaximum set for multivalued mapping F

$$H_F(x, v_1, ..., v_{s-1}, v_s^*) = \sup_{v_s}\{\langle v_s, v_s^*\rangle : v_s \in F(x, v_1, ..., v_{s-1})\},\ v_s^* \in \mathbb{R}^n,$$

$$F(x, v_1, ..., v_{s-1}; v_s^*) = \{v_s \in F(x, v_1, ..., v_{s-1}) : \langle v_s, v_s^*\rangle = H_F(x, v_1, ..., v_{s-1}, v_s^*)\}$$

respectively. For convex F we set $H_F(x, v_1, ..., v_{s-1}, v_s^*) = -\infty$ if $F(x, v_1, ..., v_{s-1}) = \varnothing$.

Let $\operatorname{int} A$ be the interior of the set $A \subset (\mathbb{R}^n)^{s+1}$ and $\operatorname{ri} A$ be the relative interior of the set A, i.e. the set of interior points of A with respect to its affine hull $\operatorname{Aff} A$.

The convex cone $K_A(v)$, $v = (x, v_1, ..., v_{s-1}, v_s)$ is called the cone of tangent directions at a point $v \in A$ to the set A if from $\bar{v} = (\bar{x}, \bar{v}_1, ..., \bar{v}_{s-1}, \bar{v}_s) \in K_A(v)$ it follows that $\bar{v}$ is a tangent vector to the set A at point $v \in A$, i.e., there exists such function $\varphi(\lambda) \in (\mathbb{R}^n)^{s+1}$ that $v + \lambda\bar{v} + \varphi(\lambda) \in A$ for sufficiently small $\lambda > 0$ and $\lambda^{-1}\varphi(\lambda) \to 0$, as $\lambda \downarrow 0$.

The cone $K_A(v)$ is called the local tent if for any $\bar{v} \in \operatorname{ri} K_A(v)$ there exists a convex cone $K \subseteq K_A(v)$ and a continuous mapping $\psi(\bar{v})$ defined in the neighbourhood of the origin such that

(1) $\bar{v} \in \operatorname{ri} K$, $\operatorname{Lin} K = \operatorname{Lin} K_A(v)$,

(2) $\psi(\bar{v}) = \bar{v} + r(\bar{v})$, $r(\bar{v})\|\bar{v}\|^{-1} \to 0$ as $\bar{v} \to 0$,

(3) $v + \psi(\bar{v}) \in A$, $\bar{v} \in K \cap S_\varepsilon(0)$ for some $\varepsilon > 0$, where $S_\varepsilon(0) \subset (\mathbb{R}^n)^{s+1}$ is the ball of radius ε. For a convex mapping F at a point $(x^0, v_1^0, ..., v_{s-1}^0, v_s^0) \in \operatorname{gph} F$

$$K_{\operatorname{gph}F}(x^0, v_1^0, ..., v_{s-1}^0, v_s^0) = \operatorname{cone}\left[\operatorname{gph} F - (x^0, v_1^0, ..., v_{s-1}^0, v_s^0)\right] = \{(\bar{x}, \bar{v}_1, ..., \bar{v}_{s-1}, \bar{v}_s):$$

$$\bar{x} = \lambda(x - x^0), \bar{v}_k = \lambda(v_k - v_k^0), k = 1, .., s\}, \forall\ (x, v_1, v_2, ... v_s) \in \operatorname{gph}F.$$

For a convex mapping F a multifunction defined by

$$F^*\left(v_s^*;(x^0,v_1^0,...,v_{s-1}^0,v_s^0)\right):=\left\{(x^*,v_1^*,...v_{s-1}^*):(x^*,v_1^*,...v_{s-1}^*,-v_s^*)\in K^*{}_{\mathrm{gph}\,F}(x^0,v_1^0,...,v_{s-1}^0,v_s^0)\right\}$$

is called a locally adjoint multifunction (LAM) to F at a point $(x^0,v_1^0,...,v_{s-1}^0,v_s^0)\in \mathrm{gph}F$, where $K^*_{\mathrm{gph}\,F}(x^0,v_1^0,...,v_s^0)$ is the dual to a cone of tangent vectors $K_{\mathrm{gph}F}(x^0,v_1^0,...,v_{s-1}^0,v_s^0)$.

The following multivalued mapping defined by

$$F^*(v^*;(x^0,v_1^0,...,v_{s-1}^0,v_s^0)):=\left\{(x^*,v_1^*,...v_{s-1}^*):H(x,v_1,v_2,...,v_{s-1},v_s^*)-H(x^0,v_1^0,...,v_{s-1}^0,v_s^*)\right\}$$
$$\leq\left\langle x^*,x-x^0\right\rangle+\sum_{k=1}^{s-1}\left\langle v_k^*,v_k-v_k^0\right\rangle,\forall(x,v_1,v_2,...,v_{s-1})\in\left(\mathbb{R}^n\right)^s\Big\},v_s\in F(x,v_1,v_2,...,v_{s-1};v_s^*)$$

is called the LAM to nonconvex mapping F at a point $(x^0,v_1^0,...,v_{s-1}^0,v_s^0)\in \mathrm{gph}F$. Clearly for the convex mapping $H(\cdot\,,v^*)$ is concave and the latter definition of LAM coincide with the previous definition of LAM. Note that, the similar notion is given by Mordukhovich [27, 28], and is called coderivative of multifunctions at a given point.

§ 2 deal with the following higher order(s -th order) discrete model labelled as (P_D):

$$\text{minimize}\quad \sum_{t=s}^{T} g(x_t,t) \tag{1}$$

$$(P_D)\; x_{t+s}\in F(x_t,x_{t+1},...,x_{t+s-1}),\; t=0,...,T-s, \tag{2}$$

$$x_t=\tilde{\alpha}_t,\; t=0,1,...,s-1 \tag{3}$$

where $x_t\in\mathbb{R}^n$, $g(\cdot,t)$ are real-valued functions, $g(\cdot,t):\mathbb{R}^n\to\mathbb{R}^1\cup\{\pm\infty\}$, F is multivalued mapping: $F:\left(\mathbb{R}^n\right)^s\to P(\mathbb{R}^n)$ and T is fixed natural numbers, $\tilde{\alpha}_t$, $t=0,1,...,s-1$ are fixed vectors. The condition

(3) is a discrete analogous of Cauchy initial conditions for higher order differential inclusions. A sequence $\{x_t\}_{t=0}^T=\{x_t: t=0,1,...,T\}$ is called a feasible trajectory for the stated problem (1) – (3).

The problem (1) – (3) is said to be convex if F and $g(\cdot,t)$ are convex multivalued function and convex proper function, respectively.

Definition 1.1 Let us say that for the convex problem (1) – (3) the regularity condition is satisfied if for points $x_t^0\in\mathbb{R}^n$, one of the following cases is fulfilled:

(i) $(x_t^0,x_{t+1}^0,...,x_{t+s}^0)\in \mathrm{ri}(\mathrm{gph}\,F)$, $x_t^0\in \mathrm{ri}(\mathrm{dom}\,g(\cdot,t)$,

(ii) $(x_t^0,x_{t+1}^0,...,x_{t+s}^0)\in \mathrm{int}(\mathrm{gph}\,F)$, $t=0,...,T-s$

(with the possible exception of one fixed t_0), and $g(\odot, t)$ are continuous at x_t^0.

Condition H .Suppose that in the problem (1) – (3) the mapping F is such that the cones of tangent directions $K_{\mathrm{gph}F}(\tilde{x}_t,...,\tilde{x}_{t+s})$ are local tents, where $\tilde{x}_t$ are the points of the optimal trajectory $\{\tilde{x}_t\}_{t=0}^T$. Suppose, moreover, that the functions $g(\cdot,t)$ admit a continuous convex upper approximation $h_t(\cdot,\tilde{x}_t)$ [3, 25, 29] at the points $\tilde{x}_t$, which ensures that the sub differentials $\partial g(\tilde{x}_t,t)=\partial h_t(0,\tilde{x}_t)$ are defined.

§ 2. Optimization of Discrete Inclusions

At the beginning we consider the convex problem (1)-(3). Let us introduce a vector $w=(x_0,x_1,...,x_T)\in\mathbb{R}^{n(T+1)}$ and define in the space $\mathbb{R}^{n(T+1)}$ the following convex sets

$$M_t=\{w=(x_0,...,x_T):(x_t,...,x_{t+s-1},x_{t+s})\in\mathrm{gph}\,F\},\ t=0,1,...,T-s,$$

$$N_t=\{w=(x_0,...,x_T):x_t=\alpha_t\},\ t=0,...,s-1.$$

First of all we should compute the cones $K_{M_t}^*(w)$, $w\in\mathrm{gph}F$.

Lemma 2.1 Let $K_{\mathrm{gph}F}(x_t,...,x_{t+s-1},x_{t+s})$, $(x_t,...,x_{t+s-1},x_{t+s})\in\mathrm{gph}F$ be cone of tangent directions. Then

$$K_{M_t}^*(w)=\left\{w^*=\left(x_0^*,...,x_T^*\right)\in K_{gph\,F}^*\left(x_t,...,x_{t++s-1},x_{t+s}\right),x_k^*=0,\ k\neq t,...,\ t+s\right\}$$

Proof. Obviously, if $w+\lambda\bar{w}\in M_t$, $t=0,...,T-s$ for sufficiently small $\lambda>0$, i.e $\left(x_t+\lambda\bar{x}_t,....,\ x_{t+s}+\lambda\bar{x}_{t+s}\right)\in\mathrm{gph}\,F$. Then $\bar{w}\in K_{M_t}(w)$. Thus

$$K_{M_t}(w)=\left\{\bar{w}:\left(\bar{x}_t,....,\ \bar{x}_{t+s}\right)\ \in K_{\mathrm{gph}F}\left(x_t,....,x_{t+s}\right)\right\}.$$

Then the proof of lemma follows immediately from the arbitrariness of components x_k, $k\neq t,....,t+s$ of vectors $\bar{w}$. Indeed on the definition of a dual cone $w^*\in K_{M_t}^*(w)$ is valid if and only if

$$\left\langle w^*,\bar{w}\right\rangle=\sum_{k=0}^{T}\left\langle x_k^*,\bar{x}_k^*\right\rangle\geq 0,\ \forall\ \bar{w}\in K_{M_t}(w).$$

But this inequality is satisfied if $x_k^*=0$, $k\neq t,...,\ t+s$. In this case it takes the form

$$\left\langle x_t^*,\bar{x}_t\right\rangle+\left\langle x_t^*,\bar{x}_t\right\rangle+\cdots+\left\langle x_{t+s}^*,\bar{x}_{t+s}\right\rangle\geq 0,$$
$$\left(\bar{x}_t,...,\bar{x}_{t+s}\right)\in K_{\mathrm{gph}\,F}\left(x_t,...,x_{t+s}\right)$$

which yields $\left(x_t^*,...,x_{t+s}^*\right)\in K_{\mathrm{gph}F}^*\left(x_t,...,x_{t+s}\right)$. This completes the proof of lemma .■

On the other hand by definition of cone of tangent vectors $w+\lambda\bar{w}\in N_t$ if and only if $\bar{x}_t=0$, $t=0,...,s-1$. As a result we have

$$K_{N_t}(w)=\left\{\bar{w}=\left(\bar{x}_0,...,\bar{x}_T\right):\bar{x}_t=0\right\},t=0,...,s-1,$$

whence

$$K^*_{N_t}(w) = \left\{ w^* = \left(x^*_0, ..., x^*_T \right) : \ x^*_k = 0, \ k \neq t \right\}, t = 0, ..., s-1. \tag{7}$$

In extended form (7) means that

$K^*_{N_0}(w) = \left\{ w^* = \left(\hat{x}^*_0, 0..., 0 \right) \right\}, ..., K^*_{N_{s-1}}(w) = \left\{ w^* = (0, ,..., 0, \hat{x}^*_{s-1}, 0..., 0) \right\}$ and so

$$\sum_{t=0}^{s-1} K^*_{N_t}(w) = \left\{ (\hat{x}^*_0, \hat{x}^*_1, ..., \hat{x}^*_{s-1}, 0, ..., 0) \right\}, \tag{8}$$

where $\hat{x}^*_t, t = 0, ..., s-1$ are arbitrary vectors.

In the sense of the terminology of first order discrete inclusions [23, 25, 31] we are ready to give the necessary and sufficient conditions for the problem (1)-(3).

Theorem 2.1 Let F be convex mapping and $g(\cdot, t)$ be convex continuous function at the points of some feasible trajectory $\left\{ x^0_t \right\}_{t=0}^T$. Then for the $\left\{ \tilde{x}_t \right\}_{t=0}^T$ to be an optimal trajectory of the problem (P_D), it is necessary that there exist a number $\lambda \in \{0,1\}$ and vectors $x^*_t, t = 0, ...T; \ \eta^{k*}_t \ (\eta^{0*}_0 \equiv 0), \eta^{k*}_{t+k}, \ t = 0, ..., T-1, \ k = 1, ..., s-1$ simultaneously not all equal to zero satisfying the discrete adjoint Euler-Lagrange inclusions and transversality conditions:

$$\left(x^*_t - \eta^{1*}_t - \eta^{2*}_t - \cdots - \eta^{s-1*}_t, \ \eta^{1*}_{t+1}, ..., \eta^{s-1*}_{t+s-1} \right) \in F^* \left(x^*_{t+s}; \left(\tilde{x}_t, \tilde{x}_{t+1}, ..., \tilde{x}_{t+s} \right) \right) - \lambda \partial g(\tilde{x}_t, t) \times \underbrace{\{0, ..., 0\}}_{s-1},$$

$$t = 0, ..., T-s, \ \partial g(\tilde{x}_t, t) \equiv \{0\}, t = 0, ..., s-1,$$

$$-x^*_t + \eta^{s-1*}_t + \eta^{s-2*}_t + \cdots + \eta^{t-T+s*}_t \in \lambda \partial g(\tilde{x}_t, t),$$

$$-x^*_T \in \lambda \partial g(\tilde{x}_T, t), \ t = T-s+1, ..., T-1,$$

Besides if the regularity condition is satisfied these conditions are sufficient for the optimality of the trajectory $\left\{ \tilde{x}_t \right\}_{t=0}^T$.

*Proof***.** Denoting $f(w) = \sum_{t=s}^{T} g(x_t, t)$ we will reduce this problem to the problem with geometric constraints. Indeed it can easily be seen that our basic problem (1) – (3) is equivalent to the following one

$$\text{minimize } f(w) \text{ subject to } \quad M = \left(\bigcap_{t=0}^{T-s} M_t \right) \bigcap \left(\bigcap_{t=0}^{s-1} N_t \right) \tag{9}$$

where M is a convex set.

Further, by the hypothesis of the theorem, $\left\{ \tilde{x}_t \right\}_{t=0}^T$ is an optimal trajectory , consequently, $\tilde{w} = (\tilde{x}_0, ..., \tilde{x}_T)$ is a solution of the problem (9). The result taken from [25, Theorem 3.4] provides necessary optimality conditions for the convex mathematical programming (9). By this theorem there exist not all zero vectors $w^*(t) \in K^*_{M_t}(\tilde{w})$, $t = 0, 1, ..., T-s$, $w^*_0(t) \in K^*_{N_t}(\tilde{w}), t = 0, ..., s-1$, and the number $\lambda \in \{0,1\}$, such that

$$\lambda w^{0*} = \sum_{t=0}^{T-s} w^*(t) + \sum_{t=0}^{s-1} w^*_0(t), \ w^0{}^* \in \partial_w f(\tilde{w}) \tag{10}$$

From definition of the function f it is easy to see that the vector $w^{0*} \in \partial_w f(\tilde{w})$ has a form

$w^{0*} = (\underbrace{0,\ldots 0}_{s}, \bar{x}_s^*, \bar{x}_{s+1}^*, \ldots, \bar{x}_{T-s}^*, \ldots, \bar{x}_T^*),\ \bar{x}_t^* \in \partial_x g(\tilde{x}_t, t).$ Furthermore according to Lemma 2.1 and formulas (7) and (8) we have

$$w^*(t) = \left(\underbrace{0,\ldots,0}_{s}, x_t^*(t), x_{t+1}^*(t), \ldots, x_{t+s}^*(t), \underbrace{0,\ldots,0}_{T-(t+s)}\right),\ \left(x_t^*(t), x_{t+1}^*(t), \ldots, x_{t+s}^*(t)\right) \in K_{\mathrm{gph}F}^*\left(\tilde{x}_t, \tilde{x}_{t+1}, \ldots, \tilde{x}_{t+s}\right),$$

$t = 0, \ldots, T-s,$

(11) $$w_0^*(t) = \left(\underbrace{0,\ldots,0}_{t}, \hat{x}_t^*, 0\ldots 0\right),\ t = 0, \ldots, s-1,$$

where $\hat{x}_t^*$, $t = 0, \ldots, s-1$ are arbitrary vectors. Now using the component wise representation of (10) for $t = 0, 1, \ldots, s-1$ we deduce that

$$0 = \hat{x}_0^* + x_0^*(0), \quad 0 = \hat{x}_1^* + x_1^*(0) + x_1^*(1),\ t = 1,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$0 = \hat{x}_{s-1}^* + x_{s-1}^*(0) + x_{s-1}^*(1) + \cdots + x_{s-1}^*(s-1),\ t = s-1.$$

Similarly for $t = s, \ldots, T-s$ we have

$$\lambda \bar{x}_{T-s+1}^* = x_{T-s+1}^*(T-2s+1) + \cdots + x_{T-s+1}^*(T-s),$$
$$\lambda \bar{x}_{T-s+2}^* = x_{T-s+2}^*(T-2s+2) + \cdots + x_{T-s+2}^*(T-s),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\lambda \bar{x}_{T-2}^* = x_{T-2}^*(T-s-2) + x_{T-2}^*(T-s-1) + x_{T-2}^*(T-s),$$
$$\lambda \bar{x}_{T-1}^* = x_{T-1}^*(T-s-1) + x_{T-1}^*(T-s),\ \lambda \bar{x}_T^* = x_T^*(T-s)$$

or more convenient form

$$\begin{aligned} &0 = \hat{x}_0^* + x_0^*(0),\ \ 0 = \hat{x}_t^* + x_t^*(0) + x_t^*(1) + \cdots + x_t^*(t), \\ &\quad t = 1, \ldots, s-1, \\ &\lambda \bar{x}_t^* = x_t^*(t-s) + x_t^*(t-s+1) + \cdots + x_t^*(t), \\ &\bar{x}_t^* \in \partial_x g(\tilde{x}_t, t), \quad t = s, \ldots, T-s. \end{aligned} \tag{12}$$

From the second formula of (11) by definition of LAM we derive that

$$\left(x_t^*(t), \ldots, x_{t+s-1}^*(t)\right) \in F^*\left(-x_{t+s}^*(t); (\tilde{x}_t, \ldots, \tilde{x}_{t+s-1}, \tilde{x}_{t+s})\right), t = 0, \ldots, T-s \tag{13}$$

Introducing the new notations

$$x_{t+k}^*(t) \equiv \eta^{k*}{}_{t+k}, \quad -x_{t+s}^*(t) \equiv x_{t+s}^*, t = 0, \ldots, T-s;\ \ k = 1, \ldots, s-1$$

by the third formula (12) we obtain

$$x_t^*(t) = \lambda \bar{x}_t^* + x_t^* - \eta^{1*}{}_t - \eta^{2*}{}_t - \cdots - \eta^{s-1*}{}_t,\ t = s, \ldots, T-s\,.$$

Then substituting these in (13) we have

$$\left(\lambda \bar{x}_t^* + x_t^* - \eta^{1*}_t - \eta^{2*}_t - \cdots - \eta^{s-1*}_t, \eta^{1*}_{t+1}, \ldots, \eta^{s-1*}_{t+s-1}\right) \in F^*\left(x^*_{t+s}; (\tilde{x}_t, \tilde{x}_{t+1}, \ldots, \tilde{x}_{t+s})\right), \ t = s, \ldots, T-s. \tag{14}$$

On the other hand it is easy to see that setting $\bar{x}_t^* \equiv 0, t = 0, \ldots, s-1$ (recall that $g(x_t, t) \equiv 0, \ t = 0, \ldots, s-1$), $\hat{x}_t^* = -x_t^*$, $t = 0, \ldots, s-1$, $\eta^{0*}_0 \equiv 0$ in the first and second equalities of (12) we can write

$$0 = -x_t^* + \eta^{t*}_t + \eta^{t-1*}_t + \cdots + \eta^{1*}_t + x_t^*(t)$$

and so taking into account arbitrariness of $\hat{x}_t^*$, $t = 0, \ldots, s-1$ we can generalize the formula (14) to the case $t = 0, 1, \ldots, s-1$. Finally, for $t = T-s+1, \ldots, T-1$ it is easy to see that

$$\begin{aligned}
&\lambda \bar{x}^*_{T-s+1} = x^*_{T-s+1}(T-2s+1) + \ldots. + x^*_{T-s+1}(T-s),\\
&\lambda \bar{x}^*_{T-s+2} = x^*_{T-s+2}(T-2s+2) + \ldots. + x^*_{T-s+2}(T-s),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\lambda \bar{x}^*_{T-2} = x^*_{T-2}(T-s-2) + x^*_{T-2}(T-s-1)\\
&+ x^*_{T-2}(T-s),\\
&\lambda \bar{x}^*_{T-1} = x^*_{T-1}(T-s-1) + x^*_{T-1}(T-s),\\
&\lambda \bar{x}^*_T = x^*_T(T-s).
\end{aligned}$$

or more generally

$$\lambda \bar{x}_t^* = x_t^*(t-s) + \ldots + x_t^*(T-s), \ \ t = T-s+1, \ldots, T-1.$$

By virtue of the accepted notation

$$\lambda \bar{x}_t^* = -x_t^* + \eta^{s-1*}_t + \eta^{s-2*}_t + \ldots + \eta^{t-T+s*}_t, t = T-s+1, \ldots, T-1. \tag{15}$$

Thus taking into account the formulas (14), (15) we complete the first part of the proof of theorem.

As for the sufficiency of the conditions obtained, it is clear that by Theorem 3.3 [25], under the regularity condition, the representation (12) holds with parameter $\lambda = 1$ for the point $w^* \in \partial_w f(\tilde{w}) \bigcap K_M^*(\tilde{w})$. ■

Theorem 2.2 Assume the condition H for the nonconvex problem (1) – (3). Then for optimality of the trajectory $\{\tilde{x}_t\}_{t=0}^T$ in the no convex problem it is necessary that there exist a number $\lambda \in \{0,1\}$ and vectors $x_t^*, t = 0, \ldots T;\ \eta^{k*}_t, \eta^{k*}_{t+k}, \ \ t = 0, \ldots, T-1, \ k = 1, \ldots, s-1$, simultaneously not all equal to zero, satisfying the conditions of Theorem 2.1.

Proof. In this case the condition H ensures the conditions of Theorem 3.24 [25] for the problem (9). Therefore, according to this theorem, we get the necessary condition as in Theorem 2.1 by starting from the relation (10), written out for the nonconvex problem. ■

Remark 2.1 The discrete adjoint Euler-Lagrange inclusions of Theorem 2.1 by using $g(x_t,t)\equiv 0,\ t=0,\dots,s-1,\ \hat{x}_t^*=-x_t^*,\ t=0,\dots,s-1,\ \eta^{0*}_0\equiv 0$ is written in more symmetrically form for all $t=0,1,\dots,T-s$. But from the arbitrariness of $\hat{x}_t^*,\ t=0,\dots,s-1$ it is clear that the first and second equations of formulas (12) are valid always. It follows that without loss of generality the conditions $\eta^{0*}_0\equiv 0,\ \partial g(x_t,t)\equiv\{0\},\ t=0,\dots,s-1$ can be removed and the adjoint s-order discrete inclusions should be justified only for $t=s,\dots,T-s$.

Remark 2.2 It is seen from the proof of the theorem that if the consideration is carried out in a separable locally convex topological space and the designation $\langle w^*,w\rangle$, is understood as the action of a linear continuous functional w^* on the element w, then from the assertion (ii) of the Definition 1.1 it is easy to conclude that the theorem is valid in this general case, too.■

In the conclusion of this section let us consider an example. At first we study the linear discrete problem

$$\text{minimize } \sum_{t=3}^{T-3} g(x_t,t)$$

$$x_{t+3}\in F(x_t,x_{t+1},x_{t+2}),\ t=0,\dots,T-3,,$$

$$x_0=\tilde{\alpha}_0,\ x_1=\tilde{\alpha}_1,\ x_2=\tilde{\alpha}_2$$

$$F(x,v_1,v_2)=\{v_3=A_0x+A_1v_1+A_2v_2+Bw: u\in U\} \tag{16}$$

where $A_i,\ i=0,1,2$ are $n\times n$ matrices, B is $n\times r$ matrix, $U\subset\mathbb{R}^r$ is a convex closed set, g is continuously differentiable function on x. It is required to find a controlling parameters $\tilde{u}_t\in U$ such that the trajectory $\{\tilde{x}_t\}_{t=0}^{T}$ corresponding to them minimizes $\sum_{t=3}^{T-1} g(x_t,t)$. In the considered case

$$F(x,v_1,v_2)=A_0x+A_1v_1+A_2v_2+BU.$$

Then by elementary computations we find that

$$F^*(v_3^*;(x,v_1,v_2))=\begin{cases}(A_0^*v_3^*,\ A_1^*v_3^*,A_2^*v_3^*), & -B^*v_3^*\in K_U^*(u),\\ \varnothing, & -B^*v_3^*\notin K_U^*(u),\end{cases} \tag{17}$$

where $v_3=A_0x+A_1v_1+A_2v_2+Bu,\ u\in U,\ A_i^*\ (i=0,1,2)$ and B^* are transposed matrices.

So using Theorem 2.1 and formula (17), for $s=3$ we get the following adjoint discrete inclusions (equations)

$$x_t^* - \eta_t^{1*} - \eta_t^{2*} = A_0^* x_{t+3}^* - \lambda g'(\tilde{x}_t, t),\ \eta_{t+1}^{1*} = A_1^* x_{t+3}^*,$$
$$\eta_{t+2}^{2*} = A_2^* x_{t+3}^*; t = 3,\dots,T-3,$$

and transversality condition

$$-x_{T-2}^* + \eta_{T-2}^{1*} + \eta_{T-2}^{2*} = \lambda g'(\tilde{x}_{T-2}, T-2),$$
$$-x_T^* + \eta_{T-1}^{2*} = \lambda g'(\tilde{x}_{T-1}, T-1),\ -x_T^* = \lambda g'(\tilde{x}_T, T).$$

Substituting $\eta_{t+1}^{1*} = A_1^* x_{t+3}^*,\ \eta_{t+2}^{2*} = A_2^* x_{t+3}^*$ into these equations and transversality condition we have

$$x_t^* = A_0^* x_{t+3}^* + A_1^* x_{t+2}^* + A_2^* x_{t+1}^* - \lambda g'(\tilde{x}_t, t),$$
$$t = 3,\dots,T-3 \tag{18}$$

$$-x_{T-2}^* + A_1^* x_T^* + A_2^* x_{T-1}^* = \lambda g'(\tilde{x}_{T-2}, T-2),$$
$$-x_T^* + A_2^* x_T^* = \lambda g'(\tilde{x}_{T-1}, T-1),\ -x_T^* = \lambda g'(\tilde{x}_T, T).$$

Besides $-B^* v_3^* \in K_U^*(u)$ means that the Weierstrass-Pontryagin maximum condition

$$< B\tilde{u}_t,\ x_t^* > = \sup_{u \in U} \langle Bu,\ x_t^* \rangle \tag{19}$$

is satisfied.

Here λ cannot be zero, because on the contrary if $\lambda = 0$, then $x_t^* \equiv 0$ for all $t = 3,\dots,T$. Thus arguing by contradiction we deduce that $\lambda = 1$ and so the conditions (18),(19) are necessary and sufficient for optimality. In other words the regularity condition in Theorem 2.1 is superfluous for linear problem and we conclude the validity of the following theorem.

Theorem 2.3. For optimality of the trajectory $\{\tilde{x}_t\}_{t=0}^T$ in problem (16) it is necessary and sufficient that there exists $\{x_t^*\}_{t=3}^T$ satisfying the adjoint Euler-Lagrange discrete inclusion (equation) (18) and maximum principle (19).

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Optimization, Volume 1: Optimality Conditions, Elements of Convex Analysis and Duality.

Alexey Izmailov and Mikhail Solodov.
Rio de Janeiro, Brazil, 2005, ISBN: 85-244-0238-5.
Second Edition -- 2009, ISBN: 978-85-244-0238-8 (In Portuguese, 270 pages).